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On the Location of the Zeros of a Polynomial

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In this paper a ring shaped region containing all the zeros of the polynomial $p(z) = a_n z^n + a_{n-1} z^{n-1} + \cdots + a_1 z + a_0$ has been obtained. Our result is best possible and sharpens some well-known results.

1. INTRODUCTION AND STATEMENT OF RESULTS

Let $p(z) = z^n + a_{n-1} z^{n-1} + \cdots + a_1 z + a_0$ be a polynomial of degree n. Then concerning a region which contains all the zeros of $p(z)$, we have the following result from Cauchy [1].

THEOREM A. *All the zeros of the complex polynomial* $p(z) = \sum_{\nu=0}^{n-1} a_\nu z^\nu + z^n$ *lie in the disc*

$$|z| \leqslant 1 + A, \tag{1.1}$$

where

$$A = \max_{0 \leqslant j \leqslant n-1} |a_j|.$$

As an improvement Joyal, *et al.* [2] proved the following theorem.

THEOREM B. *Let* $p(z) = z^n + \sum_{\nu=0}^{n-1} a_\nu z^\nu$ *be a polynomial of degree n, and let* $\beta = \max_{0 \leqslant j < n-1} |a_j|$. *Then all the zeros of* $p(z)$ *lie in the disc*

$$|z| \leqslant \tfrac{1}{2}\{1 + |a_{n-1}| + [(1 - |a_{n-1}|)^2 + 4\beta]^{1/2}\}. \tag{1.2}$$

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The expression (1.2) takes a very simple form if $a_{n-1} = 0$. If $| a_{n-1} | = 1$, it reduces to $1 + \beta^{1/2}$, which is smaller than the bound obtained in Theorem A. If $| a_{n-1} | = \beta$, Theorem B fails to give an improvement of Theorem A. In this paper we obtain a ring-shaped region containing all the zeros of $p(z)$. The outer radius of the ring is smaller than $1 + A$ even in the case when $| a_{n-1} | = \beta$. More precisely, we prove the following

THEOREM 1. *If $p(z) = z^n + a_{n-1}z^{n-1} + \cdots + a_1 z + a_0$ is a polynomial of degree n and $A = \max_{0 \leqslant j \leqslant n-1} | a_j |$, then $p(z)$ has all its zeros in the ring-shaped region*

$$\frac{| a_0 |}{2(1 + A)^{n-1} (An + 1)} \leqslant | z | \leqslant 1 + \lambda_0 A, \tag{1.3}$$

where λ_0 is the unique root of the equation $x = 1 - 1/(1 + Ax)^n$ in the interval (0, 1). *The upper bound $1 + \lambda_0 A$ in* (1.3) *is best possible and is attained for the polynomial $p(z) = z^n - A(z^{n-1} + \cdots + z + 1)$.*

If we do not wish to look for the roots of the equation $x = 1 - 1/(1 + Ax)^n$, we can still obtain a result which is an improvement of Theorem A, even in the case $| a_{n-1} | = \beta$:

THEOREM 2. *Let $p(z) = \sum_{\nu=0}^{n-1} a_\nu z^\nu + z^n$ be a polynomial of degree n and let $A = \max_{0 \leqslant j \leqslant n-1} | a_j |$. Then $p(z)$ has all its zeros in the ring-shaped region given by*

$$\frac{| a_0 |}{2(1 + A)^{n-1} (nA + 1)} \leqslant | z | \leqslant 1 + \left(1 - \frac{1}{(1 + A)^n}\right) A. \tag{1.4}$$

If we apply Theorem 1 to $z^n p(1/z)$, we get

COROLLARY 1. *Let $p(z) = 1 + \sum_{\nu=1}^{n} a_\nu z^\nu$ be a polynomial of degree n and let $A = \max_{1 \leqslant j \leqslant n} | a_j |$. Then $p(z)$ has no zero in the disc*

$$| z | < \frac{1}{1 + \lambda_0 A},$$

where λ_0 is the unique root of the equation $x = 1 - 1/(1 + Ax)^n$ in the interval (0, 1).

Similarly, on applying Theorem 2 to $z^n p(1/z)$, we get

COROLLARY 2. *If $p(z) = 1 + \sum_{\nu=1}^{n} a_\nu z^\nu$ is a polynomial of degree n and $A = \max_{1 \leqslant j \leqslant n} | a_j |$, then $p(z)$ has no zero in the disc*

$$| z | < \frac{1}{1 + (1 - 1/(1 + A)^n) A}.$$

2. Lemmas.

Lemma 1. *Let* $f(x) = x - 1 + 1/(1 + Ax)^n$, *where* n *is a positive integer and* $A > 0$. *Then if* $nA \leqslant 1$, $f(x)$ *is monotonically increasing for* $x \geqslant 0$. *If* $nA > 1$, *then there exists a* $\delta > 0$ *such that* $f(x)$ *is monotonically decreasing in the interval* $[0, \delta]$.

Proof of Lemma 1. Note that $f'(x) = 1 - nA/(1 + Ax)^{n+1}$. Hence if $nA \leqslant 1$, then $f'(x) > 0$ for $x > 0$, which implies that $f(x)$ is monotonically increasing for $x \geqslant 0$. If $nA > 1$, then $f'(0) < 0$ and hence there exists a $\delta > 0$ such that $f'(x) < 0$ in $(0, \delta)$. This completes the proof of Lemma 1.

Lemma 2. *Let* $f(x) = x - 1 + 1/(1 + Ax)^n$, *where* n *is a positive integer and* $A > 0$. *If* $nA > 1$, *then* $f(x)$ *has a unique root in the interval* $(0, 1)$.

Proof of Lemma 2.

$$\begin{aligned}(1 + Ax)^n f(x) &= (1 + Ax)^n (x - 1) + 1 \\ &= \sum_{k=0}^{n} \binom{n}{k} (Ax)^k (x - 1) + 1 \\ &= \sum_{k=1}^{n} \left[A^{k-1} \binom{n}{k-1} - A^k \binom{n}{k}\right] x^k + A^n x^{n+1} \\ &= \sum_{k=1}^{n} \frac{A^{k-1} n!}{k!\,(n - k + 1)!} [k(A + 1) - A(n + 1)]\, x^k + A^n x^{n+1}. \end{aligned} \tag{2.1}$$

Since $nA > 1$, the coefficient of x^{n+1} is positive and $k(A + 1) - A(n + 1)$ is monotonically increasing for $k \geqslant 1$, it follows from Descartes' rule of signs that $(1 + Ax)^n f(x) = 0$ has exactly one positive root. Now by Lemma 1, $f(x) < 0$ for all small, positive. Also $f(1) > 0$. Hence $f(x) = 0$ has one and only one root in $(0, 1)$ and Lemma 2 follows.

3. Proof of the Theorems

Proof of Theorem 1. First we prove that $p(z)$ has all its zeros in $|z| \leqslant 1 + \lambda_0 A$, and for this it is sufficient to consider the case when $nA > 1$ (for if $nA \leqslant 1$, then on $|z| = R > 1$, $|p(z)| \geqslant R^n - nAR^{n-1} \geqslant R^n - R^{n-1} > 0$). Following the proof of [3, Theorem (27, 2), p. 123] we get

$$| p(z)| \geqslant | z |^n \left\{1 - A \sum_{j=1}^{n} | z |^{-j}\right\}$$

$$= | z |^n - A \sum_{j=0}^{n-1} | z |^j$$

$$= | z |^n - A \frac{| z |^n - 1}{| z | - 1}. \tag{3.1}$$

Hence for every $\lambda > 0$, we have on $| z | = 1 + A\lambda$,

$$| p(z)| = (1 + A\lambda)^n - \frac{(1 + A\lambda)^n - 1}{\lambda} > 0,$$

if

$$\lambda > 1 - \frac{1}{(1 + A\lambda)^n}. \tag{3.2}$$

Thus, if λ_0 is the unique root (Lemma 2) of the equation $x = 1 - 1/(1 + Ax)^n$ (0, 1) then every $\lambda > \lambda_0$ satisfies (3.2) and hence $| p(z)| > 0$ on $| z | = 1 + A\lambda$, which implies that $p(z)$ has all its zeros in $| z | \leqslant 1 + A\lambda_0$.

Next we prove that $p(z)$ has no zero in $| z | < | a_0 |/[2(1 + A)^{n-1} (1 + nA)]$. If we denote by $g(z)$ the polynomial $(1 - z)\, p(z)$, then

$$g(z) = a_0 + \sum_{\nu=1}^{n-1} (a_\nu - a_{\nu-1})\, z^\nu + z^n - a_{n-1}z^n - z^{n+1}$$

$$= a_0 + h(z), \text{ say}.$$

If $R = 1 + A$, then

$$\max_{|z|=R} | h(z)| \leqslant R^{n+1} + R^n + | a_{n-1} | R^n + \sum_{\nu=1}^{n-1} | a_\nu - a_{\nu-1} | R^\nu$$

$$\leqslant R^n[R + 1 + A + (2n - 2) A]$$

$$= 2(1 + A)^n (nA + 1). \tag{3.3}$$

Hence on $| z | \leqslant R$,

$$| g(z)| = | a_0 + h(z)|$$

$$\geqslant | a_0 | - | h(z)|$$

$$\geqslant | a_0 | - \frac{| z |}{(1 + A)} \max_{|z|=1+A} | h(z)|, \qquad \text{by Schwarz's lemma,}$$

$$\geq |a_0| - \frac{|z|}{(1+A)}\{2(1+A)^n (nA+1)\}, \quad \text{by (3.3).}$$

$$> 0 \quad \text{if } |z| < \frac{|a_0|}{2(1+A)^{n-1}(nA+1)},$$

and the proof of Theorem 1 is complete.

We omit the proof of Theorem 2 as it follows the same lines as that of Theorem 1, noting that the inequality (3.2) is satisfied in particular (if $A > 0$) for $\lambda = 1 - 1/(1+A)^n$.

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